



**PATTERNS, SEQUENCES & SERIES (LIVE)**

**07 APRIL 2015**

**Section A: Summary Notes and Examples**

**Grade 11 Revision**

Before you begin working with grade 12 patterns, sequences and series, it is important to revise what you learnt in grade 11 about quadratic sequences. A quadratic sequence is a sequence in which the second difference is constant. The general term of this sequence is  $T_n = an^2 + bn + c$

$a$  = is a constant term which is equal to half the second difference

$b$  = constant term

$c$  = constant term

**Example**

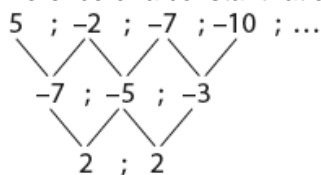
Consider the pattern: 5; -2; -7; -10; ...

1. Write down the next two terms
2. Determine an expression for the  $n^{\text{th}}$  terms
3. Show that the sequence will never have a term with a value less than -11

**Solutions**

1. -11; -10

2. Begin by identifying the sequence. Since the sequence doesn't have a common first difference or a constant ratio, we check to see if the sequence is quadratic.



$d = 2 \therefore a = 1$

To find  $b$  and  $c$  substitute  $n = 1$  into  $T_n = an^2 + bn + c$

**Equation 1**

$T_1 = 1(1)^2 + b + c$

$5 - 1 = b + c$

$\therefore 4 = b + c$

Now substitute  $n = 2$

**Equation 2**

$T_2 = 1(2)^2 + (2)b + c$

$-2 = 1(2)^2 + 2b + c$

$-6 = 2b + c$

Now solve equation 1 and 2 simultaneously

**Equation 2 minus equation 1**

$\therefore -10 = b$

$4 = -10 + c$

$\therefore 14 = c$

$\therefore T_n = n^2 - 10n + 14$





$$3. \quad \begin{aligned} n^2 - 10n + 14 &< -11 \\ n^2 - 10n + 25 &< 0 \\ (n - 5)^2 &< 0 \end{aligned}$$

This is not true for any values of  $n$  thus the sequence will not have a term less than  $-11$

### Arithmetic Sequences and Series

An arithmetic sequence or series is a linear number pattern in which the first difference is constant.

The general term formula allows you to determine any specific term of an arithmetic sequence. And the sum of formula determines the sum of a specific number of terms of an arithmetic series.

The formulae are as follows:

$$T_n = a + (n - 1)d \quad \text{where } a = \text{first term and } d = \text{constant difference}$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] \quad \text{where } a = \text{first term and } d = \text{constant difference}$$

$$S_n = \frac{n}{2}[a + l] \quad \text{where } l \text{ is the last term}$$

#### Note:

$$d = T_2 - T_1$$

$$T_1 = a \quad T_2 = a + d \quad T_3 = a + 2d \quad \text{etc.}$$

#### Example 1

The 19<sup>th</sup> term of an arithmetic sequence is 11, while the 31<sup>st</sup> term is 5.

- (a) Determine the first three terms of the sequence.

$$T_{19} = a + 18d = 11$$

$$T_{31} = a + 30d =$$

$$\therefore 12d = -6$$

$$\therefore d = -\frac{1}{2}$$

$$\therefore a + 18\left(-\frac{1}{2}\right) = 11$$

$$\therefore a = 20$$

$$\therefore 20; 19\frac{1}{2}; 19 \dots$$

- (b) Which term of the sequence is equal to  $-29$ ?

$$T_n = -29$$

$$T_n = a + (n - 1)d$$

$$20 + (n - 1)\left(-\frac{1}{2}\right) = -29$$

$$\therefore (n - 1)\left(-\frac{1}{2}\right) = -49$$

$$\therefore n - 1 = 98$$

$$\therefore n = 99$$

$$\therefore T_{99} = -29$$





Example 2

Given:  $\frac{1}{181} + \frac{2}{181} + \frac{3}{181} + \frac{4}{181} + \dots + \frac{180}{181}$

(a) Calculate the sum of the given series.

$$\frac{1}{181} + \frac{2}{181} + \frac{3}{181} + \frac{4}{181} + \dots + \frac{180}{181}$$

$$a = \frac{1}{181} \quad d = \frac{1}{181} \quad n = \frac{180}{181}$$

$$S_{180} = \frac{180}{2} \left[ 2 \left( \frac{1}{181} \right) + (179) \frac{1}{181} \right] = 90[1] = 90$$

(b) Hence calculate the sum of the following series:

$$\left( \frac{1}{2} \right) + \left( \frac{1}{3} + \frac{2}{3} \right) + \left( \frac{1}{4} + \frac{2}{4} + \frac{3}{4} \right) + \dots + \left( \frac{1}{181} + \frac{2}{181} + \dots + \frac{180}{181} \right)$$

$$\left( \frac{1}{2} \right) + \left( \frac{1}{3} + \frac{2}{3} \right) + \left( \frac{1}{4} + \frac{2}{4} + \frac{3}{4} \right) + \dots + \left( \frac{1}{181} + \frac{2}{181} + \dots + \frac{180}{181} \right)$$

$$= \frac{1}{2} + 1 + 1\frac{1}{2} + 2 \dots + 90 \quad [a = \frac{1}{2} \quad d = \frac{1}{2} \quad T_n = 90]$$

$$\therefore \frac{1}{2} + (n - 1)\frac{1}{2} = 90$$

$$\therefore 1 + n - 1 = 180$$

$$\therefore n = 180$$

$$\therefore S_{180} = \frac{180}{2} \left[ \frac{1}{2} + 90 \right] = 90 \left[ 90\frac{1}{2} \right] = 8145$$

Geometric Sequences and Series

A geometric sequence or series is an exponential number pattern in which the ratio is constant.

The general term formula allows you to determine any specific term of a geometric sequence. You have also learnt formulae to determine the sum of a specific number of terms of a geometric series.

The formulae are as follows:

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ where } r \neq 1$$

$$r = \frac{T_1}{T_2}$$

$$T_1 = a \quad T_2 = ar \quad T_3 = ar^2 \quad \text{etc.}$$





## Example 1

In a geometric sequence in which all terms are positive, the sixth term is  $\sqrt{3}$  and the eighth term is  $\sqrt{27}$ . Determine the first term and constant ratio.

$$T_6 = \sqrt{3} \text{ and } T_8 = \sqrt{27}$$

$$ar^5 = \sqrt{3}$$

$$ar^7 = \sqrt{27}$$

$$\therefore \frac{ar^7}{ar^5} = \frac{\sqrt{27}}{\sqrt{3}}$$

$$\therefore r^2 = \frac{\sqrt{27}}{\sqrt{3}}$$

$$\therefore r^2 = \sqrt{9}$$

$$\therefore r^2 = 3$$

$$\therefore r^2 = \sqrt{3} \text{ (terms are positive)}$$

$$\therefore a(\sqrt{3})^5 = \sqrt{3}$$

$$\therefore a = \frac{\sqrt{3}}{(\sqrt{3})^5}$$

$$\therefore a = \frac{1}{(\sqrt{3})^4}$$

$$\therefore a = \frac{1}{\sqrt{(3\frac{1}{2})^4}}$$

$$\therefore a = \frac{1}{9}$$

### • Convergent Geometric Series

Consider the following geometric series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \dots$$

We can work out the sum of progressive terms as follows:

$$S_1 = \frac{1}{2} = 0,5 \quad \text{(Start by adding in the first term)}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 0,75 \quad \text{(Then add the first two terms)}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 0,875 \quad \text{(Then add the first three terms)}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = 0,9375 \quad \text{(Then add the first four terms)}$$

If we continue adding progressive terms, it is clear that the decimal obtained is getting closer and closer to 1. The series is said to **converge to 1**. The number to which the series converges is called the **sum to infinity** of the series.

There is a useful formula to help us calculate the sum to infinity of a convergent geometric series.

$$\text{The formula is } S_\infty = \frac{a}{1-r}$$

If we consider the previous series  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \dots$





It is clear that  $a = \frac{1}{2}$  and  $r = \frac{1}{2}$

$$S_{\infty} = \frac{a}{1-r}$$

$$\therefore S_{\infty} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

A geometric series will converge only if the constant ratio is a number between negative one and positive one.

In other words, the sum to infinity for a given geometric series will exist only if  $-1 < r < 1$

If the constant ratio lies outside this interval, then the series will **not converge**.

For example, the geometric series  $1 + 2 + 4 + 8 + 16 + \dots$  will not converge since the sum of the progressive terms of the series diverges because  $r = 2$  which lies outside the interval  $-1 < r < 1$

### Example 1

Given the geometric series:  $8x^2 + 4x^3 + 2x^4 + \dots$

- (a) Determine the  $n^{\text{th}}$  term of the series.

$$T_n = ar^{n-1}$$

$$T_n = (8x^2) \left(\frac{1}{2}x\right)^{n-1}$$

- (b) For what value(s) of  $x$  will the series converge?

$$-1 < \frac{x}{2} < 1$$

$$= -2 < x < 2$$

- (c) Calculate the sum of the series to infinity if  $x = \frac{3}{2}$

$$S_{\infty} = \frac{a}{1-r}$$

$$\therefore S_{\infty} = \frac{8x^2}{1-\frac{x}{2}}$$

$$\therefore S_{\infty} = \frac{8\left(\frac{3}{2}\right)^2}{1-\frac{1}{2}\left(\frac{3}{2}\right)}$$

$$\therefore S_{\infty} = 72$$

### • Sigma Notation

Sigma means sum of, for example  $\sum_{n=2}^6 n + 1$  means the sum of the five terms in the sequence  $n+1$ . We determine the number of terms in this sequence by subtracting the number at the bottom, 2, from the number at the top, 6, and as seen below. There are 5 terms in the sequence.

$$\sum_{n=2}^6 n + 1 = [2 + 1] + [3 + 1] + [4 + 1] + [5 + 1] + [6 + 1]$$

$$\sum_{n=2}^6 n + 1 = 3 + 4 + 5 + 6 + 7 = 25$$





Example

(a) Calculate the value of

$$\sum_{k=1}^{100} (2k - 1)$$

$$\sum_{k=1}^{100} (2k - 1) = [2(1) - 1] + [2(2) - 1] + [2(3) - 1] + [2(4) - 1] + \dots + [2(100) - 1]$$

$$= 1 + 3 + 5 + 7 + \dots + 199$$

From the question we can see that the sequence is arithmetic and further more we have the last term therefore, we can use the formula  $S_n = \frac{n}{2} [a + l]$  to calculate the sum:

$$S_{100} = \frac{100}{2} [1 + 199]$$

$$S_{100} = 10000$$

(b) Write the following series in sigma notation:  $2 + 5 + 8 + 11 + 14 + 17$

The series is arithmetic. There are also 6 terms in the series.

$$a = 2 \quad d = 3 \quad n = 6$$

We can determine the general term as follows:

$$T_n = a + (n - 1)d$$

$$\therefore T_n = 2 + (n - 1)(3)$$

$$\therefore T_n = 2 + 3n - 3$$

$$\therefore T_n = 3n - 1$$

We can now write the series in sigma notation as follows:

$$\sum_{n=1}^6 (2k - 1)$$

## Section B: Practice Questions

### Question 1

Consider the sequence  $-2; 3; 8; 13; 18; 23; 28; 33; 38; \dots$

1.1 Determine the 100<sup>th</sup> term. (2)

1.2 Determine the sum of the first 100 terms. (2)

### Question 2

The 13<sup>th</sup> and 7<sup>th</sup> terms of an arithmetic sequence are 15 and 51 respectively.

2.1 Which term of the sequence is equal to  $-21$  (6)





### Question 3

In a geometric sequence, the 6<sup>th</sup> term is 243 and the 3<sup>rd</sup> term is 72.

Determine:

- 3.1 The constant ratio. (4)
- 3.2 The sum of the first 10 terms. (4)

### Question 4

Consider the sequence:  $\frac{1}{2}; 4; \frac{1}{4}; 7; \frac{1}{8}; 10; \dots$

- 4.1 If the pattern continues in the same way, write down the next TWO terms in the sequence. (2)
- 4.2 Calculate the sum of the first 50 terms of the sequence. (7)

### Question 5

5.1 Determine  $n$  if

$$\sum_{r=1}^n (6r - 1) = 456$$

(7)

5.2 Prove that:

$$\sum_{k=3}^n (2k - 1)n = n^3 - 4n$$

(6)

### Question 6

Consider the series

$$\sum_{n=1}^{\infty} 2\left(\frac{1}{2}x\right)^n$$

- 6.1 For which values of  $x$  will the series converge? (3)
- 6.2 If  $x = \frac{1}{2}$ , calculate the sum to infinity of this series. (3)

### Question 7

A sequence of squares, each having side 1, is drawn as shown below. The first square is shaded, and the length of the side of each shaded square is half the length of the side of the shaded square in the previous diagram.

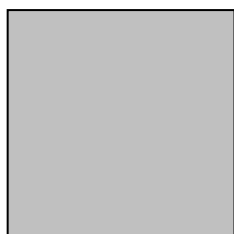


DIAGRAM 1

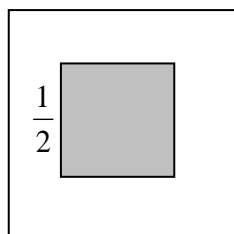


DIAGRAM 2

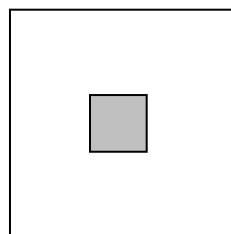


DIAGRAM 3

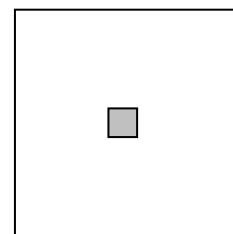


DIAGRAM 4

- 7.1 Determine the area of the unshaded region in DIAGRAM 7. (2)
- 7.2 What is the sum of the areas of the unshaded regions on the first seven squares? (5)







**Question 8**

A plant grows 1,5 m in 1<sup>st</sup> year. Its growth each year thereafter, is  $\frac{2}{3}$  of its growth in the previous year.

8.1 What is the greatest height it can reach? (3)

**Section C: Solutions**

1.1	$T_n = a + (n - 1)d$ $\therefore T_{100} = -2 + (100 - 1)(5) = 493$	$\checkmark T_n = a + (n - 1)d$ $\checkmark \therefore T_{100} = 493$ (2)
1.2	$S_n = \frac{n}{2}[2a + (n - 1)d]$ $S_{100} = \frac{100}{2}[2(-2) + (100 - 1)(5)]$ $\therefore S_{100} = 24550$	$\checkmark S_n = \frac{n}{2}[2a + (n - 1)d]$ $\checkmark T_{100} = 493$ (2)
2.1	$T_{13} = 15$ $T_7 = 51$ $\therefore a + 12d = 15 \therefore a + 6d = 51$ $\therefore a + 12d = 15 \dots A$ $\therefore a + 6d = 51 \dots B$ $\therefore 6d = -36$ $A - B$ $\therefore d = -6$ $\therefore a + 12(-6) = 15$ $\therefore a = -72 = 25$ $\therefore a = 87$  $T_n = 21$ $\therefore a + (n - 1)(d) = -21$ $\therefore 87 + (n - 1)(-6) = -21$ $\therefore 87 - 6n + 6 = -21$ $\therefore n = 19$ $\therefore T_{19} = -21$	$\checkmark a + 12d = 15$ $\checkmark a + 6d = 51$ $\checkmark d = -6$ $\checkmark a = 87$ $\checkmark 87 - 6n + 6 = -21$ $\checkmark n = 19$ (6)
3.1	$T_6 = 243$ AND $T_3 = 72$ $a.r^5 = 243 \dots A$ $a.r^2 = 72 \dots B$ $a.r^5 = 243 \dots A$ $a.r^2 = 72 \dots B$ $\therefore r^3 = \frac{27}{8} \dots A \div B$ $\therefore r = \frac{3}{2}$	$\checkmark a.r^5 = 243$ $\checkmark a.r^2 = 72$ $\checkmark r^3 = \frac{27}{8}$ $\checkmark r = \frac{3}{2}$ (4)
3.2	Substitute $r = \frac{3}{2}$ into A $\therefore a\left(\frac{3}{2}\right)^5 = 243$  $\therefore a = 32$ $\therefore S_{10} = \frac{32\left(\left(\frac{3}{2}\right)^{10} - 1\right)}{\frac{3}{2} - 1}$  $\therefore S_{10} = \frac{32\left(\left(\frac{3}{2}\right)^{10} - 1\right)}{\frac{3}{2} - 1} = 3626,5625$	$\checkmark a\left(\frac{3}{2}\right)^5 = 243$ $\checkmark a = 32$  $\checkmark \therefore S_{10} = \frac{32\left(\left(\frac{3}{2}\right)^{10} - 1\right)}{\frac{3}{2} - 1}$ $\checkmark$ answer                      (4)







4.1	$\frac{1}{16}; 13$	✓✓ answers (2)
4.2	<p><math>S_{50} = 25</math> terms of 1st sequence which is geometric +25 terms of 2nd sequence which is arithmetic .</p> $S_{50} = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ to 25 terms}\right) + (4 + 7 + 10 + 13 + \dots \text{ to 25 terms})$ $S_{50} = \frac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^{25} - 1\right)}{\frac{1}{2} - 1} + \frac{25}{2}[2(4) + 24(3)]$ <p><math>S_{50} = 0,999999 \dots + 1000</math></p> <p><math>S_{50} = 1001,00</math></p>	<p>✓ separating into an arithmetic and geometric series</p> <p>✓✓ <math>\frac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^{25} - 1\right)}{\frac{1}{2} - 1}</math></p> <p>✓ correct formulae</p> <p>✓✓ <math>\frac{25}{2}[2(4) + 24(3)]</math></p> <p>✓ answer (7)</p>
5.1	$\sum_{r=1}^n (6r - 1) = [6(1) - 1] + [6(2) - 1] + [6(3) - 1] + \dots + [6(n) - 1] = 456$ $= 5 + 11 + 17 + \dots + (6n - 1) = 456$ <p>This is an arithmetic sequence since we can see that <math>d = 6</math></p> $S_n = \frac{n}{2}(2a + (n - 1)d)$ $\therefore 456 = \frac{n}{2}(2a + (n - 1)d)$ $\therefore 456 = \frac{n}{2}(2(5) + (n - 1)6)$ $\therefore 456 = \frac{n}{2}(10 + 6n - 6)$ $\therefore 456 = \frac{n}{2}(4 + 6n)$ $\therefore 456 = 2n + 3n^2$ $\therefore 0 = 3n^2 + 2n - 456$ $\therefore (3n + 38)(n - 12) = 0$ $\therefore 3n = -38 \text{ or } n = 12$ $\therefore n = -\frac{38}{3} \text{ or } n = 12$ $\therefore n = 12$	<p>✓ expanding</p> <p>✓ correct formula</p> <p>✓ <math>456 = \frac{n}{2}(2a + (n - 1)d)</math></p> <p>✓ <math>0 = 3n^2 + 2n - 456</math></p> <p>✓ <math>(3n + 38)(n - 12) = 0</math></p> <p>✓ <math>\therefore n = -\frac{38}{3} \text{ or } n = 12</math></p> <p>✓ <math>\therefore n = 12</math></p> <p>(7)</p>
5.2	$\sum_{k=3}^n [(2k - 1)n] = 5n + 7n + 9n + \dots + [(2n - 1)n]$ <p><math>\therefore a = 5n, d = 2n</math> and number of terms = <math>2 - n</math></p>	<p>✓ expanding</p> <p>✓ <math>a = 5n, d = 2n</math></p> <p>✓ number of terms = <math>n - 2</math></p>





	$\therefore S_{n-2} = \frac{n-2}{2} [2a + (n-2-1)d]$ $\therefore S_{n-2} = \frac{n-2}{2} [2(5n) + (n-3)(2n)]$ $\therefore S_{n-2} = \frac{n-2}{2} [10n + 2n^2 - 6n]$ $\therefore S_{n-2} = \frac{n-2}{2} [2n^2 + 4n]$ $\therefore S_{n-2} = 2n(n-2) + n^2(n-2)$ $\therefore S_{n-2} = 2n^2 - 4n + n^3 - 2n^2 = n^3 - 4n$	<ul style="list-style-type: none"> <li>✓ correct formula</li> <li>✓ substitution</li> <li>✓ answer</li> </ul> <p style="text-align: right;">(6)</p>
6.1	$\sum_{n=1}^{\infty} 2\left(\frac{1}{2}x\right)^n$ $= 2\left(\frac{1}{2}x\right)^1 + 2\left(\frac{1}{2}x\right)^2 + 2\left(\frac{1}{2}x\right)^3 + 2\left(\frac{1}{2}x\right)^4 + \dots$ $= x + \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{8}x^4 + \dots$ <p>The series converges for</p> $-1 < \frac{1}{2}x < 1$ $-2 < x < 2$	<ul style="list-style-type: none"> <li>✓ <math>r = \frac{1}{2}x</math></li> <li>✓ <math>-1 &lt; \frac{1}{2}x &lt; 1</math></li> <li>✓ <math>-2 &lt; x &lt; 2</math></li> </ul> <p style="text-align: right;">(3)</p>
6.2	$a = \frac{1}{2} \quad r = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$ $\therefore S_{\infty} = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}$	<ul style="list-style-type: none"> <li>✓ <math>a</math> and <math>r</math></li> <li>✓ <math>S_{\infty}</math> formula</li> <li>✓ <math>\frac{2}{3}</math></li> </ul> <p style="text-align: right;">(3)</p>



DIAGRAM 1

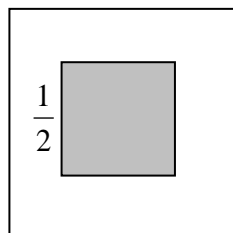


DIAGRAM 2

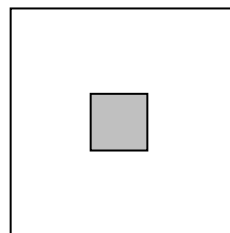


DIAGRAM 3

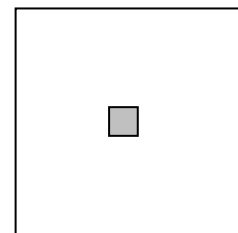


DIAGRAM 4

7.1	<p>Area of unshaded square = Area of large square – Area of small shaded square</p> $= (1)(1) - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$ $= 1 - \frac{1}{16} = \frac{15}{16}$	$\checkmark (1)(1) - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$ $\checkmark \frac{15}{16}$ <p style="text-align: right;">(2)</p>
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<p>7.2</p>	<p>Sum of the unshaded areas of the first seven squares:</p> $= (1 - 1) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{4^2}\right) + \dots + \left(1 - \frac{1}{4^6}\right)$ $= 7 - \left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots + \frac{1}{4^6}\right)$ $= 7 \left(\frac{1 - \left(\frac{1}{4}\right)^7}{1 - \frac{1}{4}}\right)$ $= 7 - 1,333251953$ $= 5,67$	<p>✓✓ Getting the pattern for the unshaded areas                  ✓ correct formula                  ✓ substitution                  ✓ answer</p> <p>(5)</p> <p>4<sup>6</sup></p>
<p>8</p>	<p><math>\therefore S_{\infty} = \frac{1,5}{1 - \left(\frac{2}{3}\right)}</math></p> <p><math>\therefore S_{\infty} = 45\text{m}</math></p> <p>Thus the greatest height is 4,5 m</p>	<p>✓ correct formula                  ✓ substitution                  ✓ answer</p>

